DETERMINATION OF UNSTEADY-STATE TEMPERATURE FIELDS IN MULTILAYERED ORTHOTOPIC PLATES

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A new hypothesis on temperature distribution with respect to the thickness of a multilayered plate is proposed. An analytical solution of the unsteady-state heat conduction problem is obtained for rectangular multilayered orthotropic plates.

To determine temperature fields in multilayered structures, different hypotheses on temperature distribution with respect to a multilayered plate are widely used. Such an approach makes it possible to reduce an initial three-dimensional problem to a two-dimensional one. Most often the hypothesis on a piecewise-linear temperature distribution is employed [2]. For a more accurate description of a temperature field, the polynomial law is used [2], while in [3] the temperature distribution with respect to a thickness is approximated by Legendre polynomials. In this case, the order of the system of resolving equations depends on the degree of the polynomial [2] or the number of layers [3]. In [4], a nonlinear law is proposed. However the aspects of a rational choice of hypothesis allowing a description of the internal thermal condition of multilayered systems are insufficiently developed.

Consider a plate composed of an arbitrary quantity ($k = 1, 2, \ldots, n$) of orthotropic layers. Contact surfaces of the layers are determined by z-coordinates of a_{k-1} and $a_k (a_k > a_{k-1})$ counted off an arbitrarily chosen coordinate plane x_1Ox_2 up to the lower and upper boundaries of a layer k. The summation is to be made with respect to dummy indices j, p, but no summation is taken over i = 1, 2; k, m. Partial derivatives of the coordinates are designated with commas on the level of subscripts, while a time derivative — by a point above the function. Superscripts, unlike exponents, are bracketed.

A heat conduction equation for the k-th layer is of the form [5]

$$c_v^{(k)}\dot{T}^{(k)} = \lambda_p^{(k)}T_{,jj}^{(k)} \quad (j = 1, 2, 3;\ p = j), \tag{1}$$

where $\lambda_p^{(k)}$ are thermal conductivities in the direction of the coordinate axes x_1, x_2 and $x_3 = z$; $c_v^{(k)}$ is the volumetric heat capacity. Between the plate layers, the following ideal thermal contact conditions are satisfied

$$T^{(k-1)}|_+ = T^{(k)}|_-;\ \lambda_3^{(k-1)}T_{,3}^{(k-1)}|_+ = \lambda_3^{(k)}T_{,3}^{(k)}|_-. \tag{2}$$

Here signs "+" and "-" designate the upper and lower boundaries of layers, respectively.

On the plate faces ($z = a_p$), the boundary conditions [5]:

of the first kind

$$T^{(1)}(x_i,\ a_0,\ \tau) = T_0(x_i,\ \tau);\ T^{(n)}(x_i,\ a_n,\ \tau) = T_n(x_i,\ \tau); \tag{3}$$

of the second kind

$$\lambda_3^{(1)}T_{,3}^{(1)}(x_i,\ a_0,\ \tau) = q_0(x_i,\ \tau);\ \lambda_3^{(n)}T_{,3}^{(n)}(x_i,\ a_n,\ \tau) = q_n(x_i,\ \tau); \tag{4}$$

of the third kind

Kiev Motor-Car and Road Institute, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 63, No. 4, pp. 486-491, October, 1992. Original article submitted July 9, 1991.

 1062-0125/92/6304-1050$12.50

$$\lambda_3^{(1)} T_{,3}^{(1)}(x_i, a_0, \tau) = \alpha_0 [T^{(1)}(x_i, a_0, \tau) - T_c^{(0)}(x_i, \tau)];$$
$$\lambda_3^{(n)} T_{,3}^{(n)}(x_i, a_n, \tau) = \alpha_n [T^{(n)}(x_i, a_n, \tau) - T_c^{(n)}(x_i, \tau)]. \quad (5)$$

may be set. In the relations (3)-(5), α_p, $T_c^{(p)}$, $q_p (p = 0, n)$ are heat transfer coefficients, ambient temperature, and heat fluxes on the plate faces, respectively. Boundary conditions, analogous to (3)-(5), may be also prescribed along the plane contour. The initial conditions for the differential equation (1) are

$$T^{(k)}(x_i, z, \tau_0) = F_0^{(k)}(x_i, z), \quad (6)$$

where $F_0^{(k)}(x_i, z)$ is the given function describing the temperature distribution over the plate at the initial moment of time $\tau = \tau_0$.

To reduce the three-dimensional heat conduction problem to the two-dimensional one based on the approach [4], a hypothesis on temperature distribution with respect to the plate thickness is built. Initially, it is assumed that the temperature distribution obeys the piecewise-linear law [1]

$$T^{(k)}(x_i, z, \tau) = \chi_p(x_i, \tau) f_p^{(k)}(z) \quad (p = 1, 2), \quad (7)$$

where $\kappa_1(x_i, \tau) = T_0(x_i, \tau)$ and $\kappa_2(x_i, \tau) = T_n(x_i, \tau)$ are the temperatures on the plate faces:

$$f_2^{(k)}(z) = \int_{a_0}^{z} [\lambda_3^{(k)}]^{-1} dz \Big/ \int_{a_0}^{a_n} [\lambda_3^{(k)}]^{-1} dz; \quad f_1^{(k)}(z) = 1 - f_2^{(k)}(z) \quad (8)$$

are the given functions of the normal.

Substitution of the law (7) into the l.h.s. of the one-dimensional unsteady-state heat conduction equation

$$c_v^{(k)} \dot{T}^{(k)} = \lambda_3^{(k)} T_{,33}^{(k)}, \quad (9)$$

subsequent integration with regard for interlayer contact conditions (2) and conditions on the plate faces (3) as well as replacement of the derivatives of temperature with respect to time by new desired functions [$T_0(x_i, \tau) = >\kappa_3(x_i, \tau)$; $T_n(x_i, \tau) = \kappa_4(x_i, \tau)$] have allowed the temperature distribution with respect to a stack thickness to be written in the form:

$$T^{(k)}(x_i, z, \tau) = \chi_p(x_i, \tau) f_p^{(k)}(z). \quad (10)$$

Henceforth $p = 1, \ldots, 4$. The functions of the temperature distribution with respect to the thickness $f^{(k)}(z)$ $(j = 3, 4)$ within each layer are the cubic parabolas and determined by the expressions

$$f_i^{(k)}(z) = d_i^{(k)} - f_2^{(k)} D_i; \quad D_i^{(k)} = d_i^{(k)}(a_n);$$
$$d_i^{(k)}(z) = \int_{a_0}^{z} [\lambda_3^{(k)}]^{-1} \left(\int_{a_0}^{z} c_v^{(k)} f_i^{(k)} dz \right) dz. \quad (11)$$

Using a variation technique, with the hypothesis (10) taken into account, we have derived a system of differential heat conduction equations which are as follows in a matrix form

$$[\mathbf{D}]\{\chi\} - [\mathbf{C}]\{\dot{\chi}\} = \{\mathbf{q}\}, \quad (12)$$

where [**D**] is the matrix of differential operators, the elements of which are

$$d_{jp} = P_1^{(jp)}(\ldots)_{,11} + P_2^{(jp)}(\ldots)_{,22} - P_3^{(jp)}(\ldots) \quad (j = 1, \ldots, 4);$$
$$d_{11} = d_{11} - \alpha_0(\ldots); \quad d_{22} = d_{22} - \alpha_n(\ldots); \quad (13)$$

TABLE 1. Temperature Distribution with Respect to Thickness at the Center of a Three-layered Orthotropic Plate

Time, sec	z, m	Proposed solution		Hypothesis (7)		Hypothesis [4]		Three-dimensional solution
		T, °C	Δ, %	T, °C	Δ, %	T, °C	Δ, %	T, °C
300	—0,02	43,04	0,9	36,35	16,3	42,98	1,0	43,41
	—0,01	19,87	—0,6	25,26	—27,8	19,87	—0,6	19,76
	0,01	2,86	—4,4	3,07	—12,0	2,92	—6,6	2,74
	0,02	2,85	—9,2	—8,02	407	2,73	—4,6	2,61
500	—0,02	50,72	1,0	46,04	10,1	50,73	0,9	51,21
	—0,01	28,76	—0,1	33,80	—17,6	28,77	—0,1	28,73
	0,01	7,73	—3,5	9,33	—24,9	7,77	—4,0	7,47
	0,02	6,39	—4,2	—2,91	147	6,39	—4,2	6,13
1000	—0,02	62,19	0,7	59,28	5,4	62,19	0,7	62,63
	—0,01	43,10	0,1	47,80	—10,9	43,11	0,0	43,11
	0,01	21,26	—2,0	24,85	—19,2	21,27	—2,1	20,84
	0,02	18,69	—2,5	13,37	26,7	18,69	—2,5	18,24
3000	—0,02	82,91	0,3	82,79	0,4	82,91	0,3	83,14
	—0,01	70,46	—0,1	74,80	—6,3	70,46	—0,1	70,39
	0,01	54,25	—0,8	58,59	—8,9	54,26	—0,8	53,81
	0,02	50,55	—1,0	50,49	—0,9	50,55	—1,0	50,03
∞	—0,02	94,40	0,0	94,15	0,3	94,40	0,0	94,40
	—0,01	85,70	0,0	87,68	—2,3	85,70	0,0	85,70
	0,01	72,95	0,0	74,76	—2,5	72,95	0,0	72,95
	0,02	68,67	0,0	68,29	0,6	68,67	0,0	68,67

[C] is the matrix characterizing heat capacity of the system; ($c_{jp} = R_{jp}$); $\{\chi\} = \{\chi_p\}$ is the vector of the desired functions;

$$\{q\} = \{q_0 - \alpha_0 T_c^{(0)};\ q_n - \alpha_n T_c^{(n)};\ 0;\ 0\} \tag{14}$$

is the vector of thermal impact.

From the contour integral of the variational equation we obtain the corresponding boundary conditions

$$(P_m^{(pj)} \chi_{j,m} + Q_p + G_{pj}\chi_j - Y_p T_c^{(T)})\,\delta\chi_p = 0 \ (j = 1,\ \ldots,\ 4), \tag{15}$$

where m is the normal to the contour; $T_c^{(T)}$ is the ambient temperature at the boundary with the end faces of the plate.

In (13), (15), the generalized integrated thermophysical characteristics of a multilayered plate are introduced to describe heat transfer by conduction:

$$P_i^{(jp)} = \int_{a_0}^{a_n} \lambda_i^{(k)} f_j^{(k)} f_p^{(k)} dz;\ P_3^{(jp)} = \int_{a_0}^{a_n} \lambda_3^{(k)} f_{j,3}^{(k)} f_{p,3}^{(k)} dz; \tag{16}$$

thermal inertia

$$R_{jp} = \int_{a_0}^{a_n} c_v^{(k)} f_j^{(k)} f_p^{(k)} dz, \tag{17}$$

as well as the boundary conditions at the end faces of the plate:

$$Q_j = \int_{a_0}^{a_n} q_T^{(k)} f_j^{(k)} dz;\ Y_j = \int_{a_0}^{a_n} \alpha_T^{(k)} f_j^{(k)} dz; \tag{18}$$

$$G_{jp} = \int_{a_0}^{a_n} \alpha_T^{(k)} f_j^{(k)} f_p^{(k)} dz \ (j = 1,\ \ldots,\ 4). \tag{19}$$

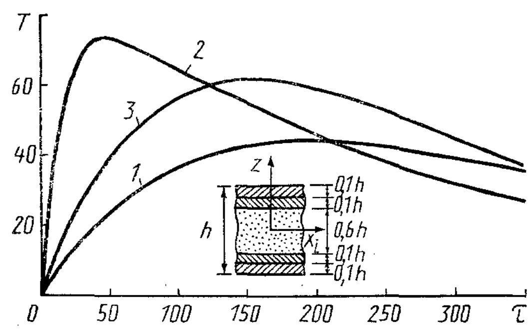

Fig. 1. Temperature distribution in time at the central point of the five-layered orthotropic plate. T, °C; τ, sec.

In (18) and (19), $\alpha_T^{(k)}$ and $q_T^{(k)}$ are the heat transfer coefficients and heat fluxes across the end faces of the plate.

The overall order of the system (12) does not depend on the number of layers and is equal to eight. On each edge of the plate, four boundary conditions (15) at once must be satisfied.

Consider an analytical solution of the system (12) for a rectangular ($a_1 \times a_2$) layered plate, along the contour of which a zero temperature is maintained.

On the plate faces, the following thermal impact conditions may be prescribed

$$q_j(x_i, \tau) = \bar{q}_i(\tau) \sum_r \sum_s u_{rs}^{(j)} \sin(\alpha x_1) \sin(\beta x_2);$$
$$T_c^{(j)}(x_i, \tau) = \bar{T}_c^{(j)}(\tau) \sum_r \sum_s v_{rs}^{(j)} \sin(\alpha x_1) \sin(\beta x_2) \quad (j = 0,\ n), \tag{20}$$

where $\alpha = r\pi/a_1$; $\beta = s\pi/a_2$; $\bar{q}_j$, $\bar{T}_c^{(j)}$ are the given functions of time; $u_{rs}^{(j)}$, $v_{rs}^{(j)}$ are the Fourier coefficients for series expansion of thermal impacts.

To satisfy the conditions along the contour, the distribution of the desired functions over the plate is represented in the following form:

$$\chi_p(x_i, \tau) = \sum_r \sum_s A_{rs}^{(p)}(\tau) \sin(\alpha x_1) \sin(\beta x_2) \quad (r,\ s = 1,\ 3,\ 5,\ \ldots). \tag{21}$$

Substitution of (20) and (21) into the system (12) yields a sequence of the systems of ordinary first-order differential equations with constant coefficients for the given pairs of r and s values:

$$[H]_{rs}\{A\}_{rs} + [C]_{rs}\{\dot{A}\}_{rs} = \{q\}_{rs}. \tag{22}$$

Here $\{A\}_{rs} = \{A_{rs}^{(p)}(\tau)\}$ is the vector of unknown coefficients; $[H]_{rs}$, $[C]_{rs}$ are the heat conduction and heat capacity matrices obtained, according to (13), with regard for differentiation:

$$\{q\}_{rs} = \{-\bar{q}_0 u_{rs}^{(0)} + \alpha_0 \bar{T}_c^{(0)} v_{rs}^{(0)},\ -\bar{q}_n u_{rs}^{n} + \alpha_n \bar{T}_c^{(n)} v_{rs}^{(n)},\ 0,\ 0\} \tag{23}$$

is the vector of the r.h.s.

Following [6], a solution of the system (22) is sought in the form

$$\{A\}_{rs} = \{e\}_p y_p, \tag{24}$$

where $\{e\}_p$ are the eigenvectors of the equation $(\lambda[C] + [H])\{e\} = 0$; $y_p = y_p(\tau)$ are some new desired functions. The matrices [H] and [C] are the positive definite matrices and, consequently, all eigenvalues of λ_p will be real and negative.

Substituting (24) into the system (22) and multiplying both sides by the corresponding eigenvalues with allowance for their orthogonality, we arrive at the system of independent equations

$$\dot{y}_p - \lambda_j y_p = \{e\}_p\{q\} \quad (j = p). \tag{25}$$

The latter equations may be solved by traditional methods [7]. Integration constants are found from the initial conditions.

The desired functions are determined according to (21) with account of (24). Then from (10) the temperature distribution with respect to the thickness is found at an arbitrary point of the k-th layer of the plate.

Consider some examples illustrating the proposed approach.

Example 1. An unsteady-state temperature field is to be determined in a three-layered ($a = a_i = 0.4$ m) plate at thermal impact, constant in time. On the plate contour, a zero temperature is maintained. The ambient temperature at the boundary with the plate faces varies according to the law

$$T_c^{(j)} = \bar{T}_c^{(j)} \sin(\pi x_1/a) \sin(\pi x_2/a) \quad (j = 0,\ n), \tag{26}$$

where $\bar{T}_c^{(0)} = 20°C$, $\bar{T}_c^{(n)} = 120°C$. Heat transfer coefficients are $(\alpha_0, \alpha_n) = (10; 50)$ W/(m·K). The initial temperature distribution is assumed to be zero. Each layer represents a unidirectional-reinforced graphite-epoxy composite with the following characteristics: $(\lambda_1, \lambda_2 = \lambda_3) = (2.7; 1.34)$ W/(m·K), its volumetric heat capacity is $c_v = 2.7 \times 10^6$ J/(m^3·K). The layers are directed in turn at the angles 0 and 90^0 to the x_1 axis, the thickness of the layers is as follows: $h_k = (0.25; 0.5; 0.25)h$ (k = 1, 2, 3), where h = 0.04 m is the total thickness of the plate.

Results of solution of the given problem are listed in Table 1. As a standard solution, the three-dimensional finite-element solution is taken. As is seen, the results obtained on the basis of the suggested model and the nonlinear law [4] are close and consistent with the three-dimensional solution. The piecewise-linear law allows reliable results to be obtained only for the moments of time, to which a temperature field distribution, close to a stationary one, corresponds.

Example 2. A nonstationary temperature field in a five-layered plate ($a = a_i = 1.0$ m) is exposed to time-variable thermal impact. The conditions along the contour and the initial temperature distribution are analogous to those in the previous example. A heat flux is supplied to a lower surface according to the law $q_0 = \bar{q}_0 \exp(-\omega\tau)$ where $\bar{q}_0 = 100$ W/m^2. On the upper surface, heat transfer by convection proceeds. The heat transfer coefficient is $a_n = 20$ W/(m^2·K), and the ambient temperature varies as $T^{(n)} = \bar{T}_c^{(n)}\exp(-\omega\tau)$, where $\bar{T}_c^{(n)} = 200°C$, $\omega = 0.0075$. A middle layer of the plate (k = 3) is fabricated from superlight foam plastic with the following characteristics: $\lambda_1 = \lambda_2 = 0.04$ W/(m·K), $c_v = 4170$ J/(m^3·K). The remaining layers are made of glass-like transparent plastic. The characteristics of outer layers (k = 1.5) are as follows: $(\lambda_1, \lambda_2 = \lambda_3) = (0.672; 0.229)$ W/(m·K), $c_v = c_v = 1.875 \times 10^5$ J/(m^3·K). The layers k = 2, 4 are directed at an angle of 90° with respect to the outer layers. A thickness of the layers is $h_k = (0.01; 0.01; 0.06; 0.01; 0.01)$ m (k = 1...5).

Figure 1 represents the temperature distribution in time at the center of the plate at the upper layer boundaries (z = 0.4 h and z = 0.5 h) obtained with the help of the proposed model (curves 1 and 2). Temperatures at the same points corresponding to law (7) are practically equal and therefore they are indicated by curve 3.

Analysis of the results has shown that the proposed model allows a description of the thermal condition of layered structures practically with the same accuracy as the model [4] does. For numerical realization, it is reasonable to employ the method of finite elements. Relations (10) of the model include a smaller number of desired functions (four instead of six), as compared to [4], that leads, correspondingly, to systems of resolving equations of a smaller order and considerably expands the range of problems to be solved.

NOTATION

λ_j (j = 1, 2, 3), thermal conductivities of the orthotopic body; c_v, volumetric heat capacity; $T^k(x_i, z, \tau)$, temperature at an arbitrary point of the k-th layer; q_p, α_p, $T_c^{(p)}$, heat flux, heat transfer coefficient, and ambient temperature on the lower (p = 0) and upper (p = n) plate surfaces, respectively.

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